MATH 124B: HOMEWORK 1

Suggested due date: August 8th, 2016

(1) For $m, n \in \mathbb{Z}$, compute the following integrals

(a) $\displaystyle\int_{-L}^{L} \cos\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)dx.$

(b) $\displaystyle\int_{-L}^{L} \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)dx.$

(2) Let $\phi(x) = x^2$ for $0 \leq x \leq 1$. Calculate its Fourier sine and cosine series.

(3) Find the Fourier cosine series of the function $|\sin x|$ in the interval $(-\pi, \pi)$. Use it to find the sums

$$\sum_{n=1}^{\infty} \frac{1}{4n^2-1}, \qquad \text{and} \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1}.$$

(4) Solve the **Neumann** initial value problem

$$\begin{cases} u_{tt} = c^2 u_{xx}, & \text{on } 0 < x < \pi \\ u_x(0,t) = 0 = u_x(\pi,t) \\ u(x,0) = 0 \\ u_t(x,0) = \cos^2(x). \end{cases}$$

(5) Show that $\cos(x) + \cos(ax)$ is periodic if a is a rational number. What is its period?

(6) How does differentiation and integration affect the even-odd property of a function? For integration, assume the constant of integration is 0.

(7) Let $\phi(x)$ be a function of period π. If $\phi(x) = \sum_{n=1} a_n \sin(nx)$ for all x, find the odd coefficients.

Solutions

(1) Directly compute using the identities

$$2\sin(A)\cos(B) = \sin(A+B) + \sin(A-B)$$
$$2\cos(A)\cos(B) = \cos(A+B) + \cos(A-B).$$

(2) Directly applying the formulas, we have for $n \neq 0$

$$A_n = 2\int_0^1 x^2 \cos(n\pi x)dx = \frac{(-1)^n 4}{n^2\pi^2}.$$

and

$$A_0 = 2\int_0^1 x^2 dx = \frac{2}{3}.$$

Similar computation for the sine series.

(3) Using the cosine coefficient formula for the symmetric interval, we have

$$\begin{aligned} A_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin x| \cos(nx) dx \\ &= \frac{2}{\pi}\int_0^{\pi} \sin(x)\cos(nx) dx \\ &= \begin{cases} -\frac{4}{\pi(n^2-1)} & n \text{ even} \\ 0 & n \text{ odd .} \end{cases} \end{aligned}$$

Hence

$$|\sin(x)| = \frac{2}{\pi} + \sum_{n=1}^{\infty} -\frac{4}{\pi(4n^2-1)} \cos(2nx).$$

Let $x = 0$, then

$$\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2}$$

and when $x = \frac{\pi}{2}$,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1} = \frac{1}{2} - \frac{\pi}{4}$$

(4) The general formula is given, to compute the cosine series for $\cos^2(x)$, one simply uses the identity $\cos^2(x) = \frac{1+\cos(2x)}{2}$.

(5) Let $\alpha = \frac{p}{q}$, $\gcd(p, q) = 1$. The period is given by $2\pi q$.

(6) Let $f(x)$ be even. Its derivative is given by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

So

$$\begin{aligned} f'(-x) &= \lim_{h\to 0} \frac{f(-x+h) - f(-x)}{h} \\ &= \lim_{h\to 0} -\frac{f(x-h) - f(x)}{-h} = -f'(x). \end{aligned}$$

So the derivative is an odd function. The other case is similar. For antiderivatives, we have (assuming the constant of integration is 0,

$$F(x) = \int_a^x f(s) ds.$$

Then changing variables $s = -t$

$$\begin{aligned} F(-x) &= F(a) + \int_a^{-x} f(s)ds \\ &= F(a) + -\int_{-a}^{x} f(-t)dt \\ &= -F(x) \end{aligned}$$

So the antiderivative is odd. The other case is similar.

(7) Use $\sin(n(x+\pi)) = (-1)^n \sin(nx)$ and periodicity to obtain $a_n = 0$ for n odd.